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PERTURBATIONS ABOUT STRONG SPHERICAL SHOCK WAVES

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Foreword

G. I. Taylor found a solution of the gas dynamic equations which describes the pressure waves produced by an explosion. This description can be used at distances from the explosive large compared to the explosive dimensions, but small enough that the shock pressure is large compared to atmospheric pressure. In fact the explosive is assumed to be a point, and the atmospheric pressure is neglected compared to the shock pressure, in obtaining the solution. For these reasons, the solution is called either the "point blast" or "strong shock" solution. The same solution was also obtained by J. von Neumann, and a solution of the same type was found for explosions in water by H. Primakoff. Similar solutions were studied by G. Guderley in Germany.

The two opposing restrictions on the range of validity of the Taylor "point blast" solution are such that, for ordinary explosives, there is practically no range in which they are both satisfied. For nuclear explosives, however, the size of the explosive is so small that there is a range in which this solution is useful. This has been demonstrated by comparing this solution with experimental results.

In order to extend the solution to greater ranges, where the shock is weaker, it is necessary to take account of the atmospheric pressure ahead of the shock. The present report by Dr. Morawetz attempts to do this by determining a correction to the "point blast" solution, which correction is due to the previously neglected atmospheric pressure. A system of linear equations is obtained for the correction and their solution is analyzed. Finally for the case of Primakoff's "point blast" in water, the solution is given explicitly and graphs of various quantities are given for both the original and the improved solutions.

It cannot be expected that the range of validity of the "point blast" solution can be extended very much by such a perturbation method. However the qualitative and quantitative nature of the corrections can give an indication of the range at which the solution begins to fail, and can also indicate the manner in which the solution changes due to atmospheric pressure.

Joseph B. Keller

PERTURBATIONS ABOUT STRONG SPHERICAL SHOCK WAVES

Introduction.

It has been shown by G. I. Taylor* and others that there are solutions of the equations of spherical flow of the form

$$u = \lambda^{-1} \frac{r}{t} U_o(r^{-\lambda}t)$$

$$c = \lambda^{-1} \frac{r}{t} C_o(r^{-\lambda}t)$$

$$p = \lambda^{-2} \frac{r^2}{t^2} P_o(r^{-\lambda}t)$$

where λ is any constant and U_o, C_o, P_o satisfy nonlinear ordinary differential equations. These solutions, however, can represent a flow behind a shock only if we can neglect the pressure ahead of the shock. Here we shall study the first order effects of this pressure and the modifications produced on the original flow.

The problem of finding the flow behind an expanding shock wave of constant energy is reduced to solving some ordinary differential equations which depend only on γ, the ratio of the specific heats. In the case of $\gamma = 7$, these equations can be solved and the perturbations in the flow quantities expressed explicitly.

The difference between the flow behind a strong shock and a strong detonation can also be studied in the same way.

The Equations of Motion.

The equations for spherically symmetric flow are:

$$u_t + uu_r + \frac{1}{\rho} p_r = 0 \quad ,$$

$$(1) \qquad \rho_t + u\rho_r + \rho(u_r + \frac{2u}{r}) = 0 \quad ,$$

$$(p\rho^{-\gamma})_t + u(p\rho^{-\gamma})_r = 0 \quad ,$$

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* G. I. Taylor: The formation of a blast wave by a very intense explosion. Proceedings of the Royal Society, Series A, Vol. 201, March 1950, p. 159.

where u is the radial velocity, p is the pressure, ρ is the density and γ is the ratio of the specific heats of the gas, while r is the distance from the origin and t is time.

If we introduce the variables

$$(2) \qquad \eta = r^{-\lambda}t$$

$$(3) \qquad U = \lambda t r^{-1} u$$

$$C = \lambda t r^{-1} c = \lambda t r^{-1}\sqrt{\frac{\gamma p}{\rho}}$$

$$P = \lambda^2 t^2 r^{-2} p \quad ,$$

where λ is any positive constant, we obtain, using

$$(4) \qquad \frac{d\eta}{\eta} = -\lambda\frac{dr}{r} + \frac{dt}{t}$$

in (1),

$$\eta U_\eta(1-U) + UtU_t - \gamma^{-1}C^2\frac{\eta P_\eta}{P} + U(\alpha U-1) + 2\alpha\gamma^{-1}C^2 = 0$$

$$(5) \quad -\eta U_\eta + \frac{\eta P_\eta}{P}(1-U) - 2\frac{\eta C_\eta}{C}(1-U) + U\frac{tP_t}{P} - 2U\frac{tC_t}{C} + 3\alpha U = 0$$

$$\frac{\eta P_\eta}{P}(1-U) - \frac{2\gamma}{\gamma-1}(1-U)\frac{\eta C_\eta}{C} + U\frac{tP_t}{P} - \frac{2\gamma}{\gamma-1}U\frac{tC_t}{C} + \frac{2}{\gamma-1}(1-\alpha U) = 0$$

where $\alpha = \lambda^{-1}$.

Equations (5) may be rewritten as

$$D\eta U_\eta = A - (1-U)tU_t - \gamma^{-1}C^2\frac{tP_t}{P}$$

$$(6) \qquad D\frac{\eta C_\eta}{C} = B - \frac{\gamma-1}{2}tU_t - \frac{D}{1-U}\frac{tC_t}{C} - \frac{(\gamma-1)}{2\gamma}\frac{C^2}{1-U}\frac{tP_t}{P}$$

$$D\frac{\eta P_\eta}{P} = E - \gamma tU_t - (1-U)\frac{tP_t}{P}$$

where

$$A = (1-U)(1-aU)U - (3aU-2\gamma^{-1}(1-a))C^2$$

$$B = (1-U)(1-aU)-(\gamma-1)[(1-aU) - \tfrac{3}{2}(1-a)]U - [a+\gamma^{-1}(1-a)(1-U)^{-1}]C^2$$

(7)

$$E = (1-U)[2(1+\gamma)(1-aU) - 3\gamma] + \gamma(1-aU) - 2aC^2$$

$$D = (1-U)^2 - C^2 \quad .$$

There are two identities relating A, B, D and E,

$$-A + \frac{2}{\gamma-1} B(1-U) + D(3aU - \frac{2}{\gamma-1}(1-aU)) = 0$$

(8)

$$-A + \gamma^{-1} E(1-U) + D(3aU - \frac{2}{\gamma}(1-aU)) = 0 \quad .$$

The special solutions for spherical flow known as spherical "progressing" waves are solutions of (6) that are independent of t. They represent states behind or ahead of shocks and detonations if the shock or detonation front is given by η = constant. We represent any such flow by $U_o(\eta)$, $C_o(\eta)$, $P_o(\eta)$. Then from (6), we find the following nonlinear ordinary differential equations for U_o, C_o and P_o,

$$D_o \eta U_{o\eta} = A_o$$

(9)

$$D_o \eta \frac{C_{o\eta}}{C_o} = B_o$$

$$D_o \eta \frac{P_{o\eta}}{P_o} = E_o$$

where $A_o = A(U_o, C_o)$ etc.

From the first two equations of (6) we obtain an equation for C_o as a function of U_o,

$$\frac{dC_o}{dU_o} = \frac{B_o C_o}{A_o} \quad .$$

The solutions of this equation then yield, through (6), P_o and η as functions of U_o.

We shall consider here flows that are approximately progressing waves, that is flows given by

$$(10)\qquad \begin{aligned} U &= U_o(\eta) + \varepsilon U_1(\eta, t) + O(\varepsilon^2) \\ C &= C_o(\eta)(1 + \varepsilon C_1(\eta, t) + O(\varepsilon^2)) \\ P &= P_o(\eta)(1 + \varepsilon P_1(\eta, t) + O(\varepsilon^2)) \end{aligned}$$

where ε is some small parameter.

Substituting (10) in (6) and equating first order terms in ε we find

$$(11)\qquad \begin{aligned} \eta U_{1\eta} &= \left(\frac{A}{D}\right)_U^o U_1 + \left(\frac{A}{D}\right)_C^o C_o C_1 - D_o^{-1}(1-U_o)tU_{1t} - \gamma^{-1} C_o^2 D_o^{-1} tP_{1t} \\ \eta C_{1\eta} &= \left(\frac{B}{D}\right)_U^o U_1 + \left(\frac{B}{D}\right)_C^o C_o C_1 - \frac{\gamma-1}{2} D_o^{-1} tU_{1t} \\ &\quad - \frac{D_o^{-1}}{1-U_o}\frac{tC_{1t}}{C_o} - \frac{\gamma-1}{2\gamma}\frac{C_o^2}{1-U_o} D_o^{-1} tP_{1t} \\ \eta P_{1\eta} &= \left(\frac{E}{D}\right)_U^o U_1 + \left(\frac{E}{D}\right)_C^o C_o C_1 - \gamma D_o^{-1} tU_{1t} - (1-U_o) D_o^{-1} tP_{1t} \quad . \end{aligned}$$

Here $\left(\frac{A}{D}\right)_U^o = \left(\frac{\partial}{\partial U}\frac{A}{D}\right)_{U=U_o(\eta),\, C=C_o(\eta)}$.

Conditions at a Shock.

The boundary conditions on the flow quantities at a shock are given by

$$(12)\qquad \begin{aligned} \rho(u-z) &= -\rho_1 z \\ \rho(u-z)^2 + p &= \rho_1 z^2 + p_1 \\ \frac{\gamma p}{\rho(\gamma-1)} + \frac{1}{2}(u-z)^2 &= \frac{z^2}{2} + \frac{\gamma p_1}{\rho_1(\gamma-1)} \end{aligned}$$

where z is the velocity of the shock front, and ρ_1, p_1 are the density and pressure ahead of the front.

In terms of the variables U, C, P of (3) equations (12) become

$$\rho(U-Z) = -\rho_1 Z$$

(13) $$\rho(U-Z)^2 + P = \rho_1 Z^2 + \lambda^2 \eta^{2\alpha} t^{2-2\alpha} p_1$$

$$\mu^2(U-Z)^2 + (1-\mu^2)C^2 = \mu^2 Z^2 + (1-\mu^2)c_1^2\lambda^2 \eta^{2\alpha} t^{2-2\alpha}$$

where

(14) $$Z = \lambda t r^{-1} z$$

(15) $$\mu^2 = (\gamma-1)(\gamma+1)^{-1}$$

$$c_1^2 = \gamma p_1 \rho_1^{-1} \quad .$$

From (13) we see that a spherical progressing wave can represent the state behind a shock for $\lambda \neq 1$ only if $c_1 = p_1 = 0$, that is, only if the shock is infinitely strong. In this case the position of the front is given by

(17) $$\eta = H_o$$

where H_o is some constant and thus from (14)

(18) $$Z = \lambda t r^{-1} \frac{dr}{dt} = 1 \quad .$$

We assume now that the deviation in the shock path is small, and set

(19) $$\eta = H_o(1 + \varepsilon H_1(t)) \quad \text{on the shock} \quad .$$

Then from (14) we find to first order in ε

(20) $$Z = 1 - \varepsilon t H_{1t} \quad .$$

Substituting (20) in (13) and setting

(21) $$\varepsilon = \lambda^2 c_1^2 H_o^{2\alpha}$$

we obtain

$$U = (1-\mu^2)(1 - \varepsilon t H_{1t} - \varepsilon r^{2\lambda-2})$$

(22) $$c^2 = \mu^2(1+\mu^2)(1 - 2\varepsilon a r H_{1r} + \varepsilon \frac{1-2\mu^4}{\mu^2(1+\mu^2)} t^{2-2a})$$

$$P = \rho_1(1-\mu^2)(1 - 2\varepsilon t H_{1t} - \varepsilon \frac{\mu^2}{1+\mu^2} t^{2-2a})$$

for $\eta = H_o(1+\varepsilon H_1(t))$.

On the other hand, expanding $U(H_o(1+\varepsilon H_1(t),t,\varepsilon)$ etc. in powers of ε we find

(23) $$U(H_o(1+\varepsilon H_1(t)),t,\varepsilon)$$
$$= U_o\Big(H_o(1+\varepsilon H_1(t))\Big) + \varepsilon U_1(H_o(1+\varepsilon H_1(t)),t) + O(\varepsilon^2)$$
$$= U_o(H_o) + \varepsilon U_1(H_o,t) + \varepsilon H_1 H_o \left(\frac{\partial U_o}{\partial \eta}\right)_{\eta=H_o} H_1(t) + O(\varepsilon^2)$$
$$= U_o(H_o) + \varepsilon U_1(H_o,t) + \varepsilon \left(\frac{A_o}{D_o}\right)_{\eta=H_o} H_1(t) + O(\varepsilon^2)$$

$$C(H_o(1+\varepsilon H_1(t)),t,\varepsilon)$$
$$= C_o(H_o)\left[1 + \varepsilon C_1(H_o,t) + \varepsilon \left(\frac{B_o}{D_o}\right)_{\eta=H_o} H_1(t) + O(\varepsilon^2)\right]$$

$$P(H_o(1+\varepsilon H_1(t)),t,\varepsilon)$$
$$= P_o(H_o)\left[1 + \varepsilon P_1(H_o,t) + \varepsilon \left(\frac{E_o}{D_o}\right)_{\eta=H_o} H_1(t) + O(\varepsilon^2)\right] .$$

Comparing (23) with (22) we have

(24) $$U_o(H_o) = 1 - \mu^2$$
$$c_o^2(H_o) = \mu^2(1+\mu^2)$$
$$P_o(H_o) = \rho_1(1-\mu^2) .$$

It is then clear from (9) that U_o, C_o and P_o/ρ_1 are all functions of η/H_o which depend only on γ.

From the first order terms in ε, we find

$$U_1(H_o,t) = -\left(\frac{A_o}{D_o}\right)_{\eta=H_o} H_1(t) - (1-\mu^2)tH_{1t}(t) - (1-\mu^2)t^{2-2a}$$

$$(25) \qquad C_1(H_o,t) = -\left(\frac{B_o}{D_o}\right)_{\eta=H_o} H_1(t) - tH_{1t}(t) + \frac{1-2\mu^4}{2\mu^2(1+\mu^2)} t^{2-2a}$$

$$P_1(H_o,t) = -\left(\frac{E_o}{D_o}\right)_{\eta=H_o} H_1(t) - 2tH_{1t}(t) - \frac{\mu^2}{1+\mu^2} t^{2-2a} \quad .$$

The perturbation $H_1(t)$ in the position of the shock can be eliminated and (25) reduced to two conditions on the flow quantities, namely for $\eta = H_o$,

$$(26) \qquad \begin{aligned} &a_1U_1 + \beta_1C_1 + \gamma_1P_1 = \delta_1t^{2-2a} \\ &a_2U_1 + \beta_2C_1 + \gamma_2P_1 + a_{21}tU_{1t} + \beta_{21}tC_{1t} = \delta_2t^{2-2a} \end{aligned}$$

where a_1, β_1, etc. are all constants.

To find the flow behind a shock we now have to solve the system of hyperbolic differential equations (11) of third order and two conditions (26) on a space-like line, the shock path. One more condition must be prescribed. For example, we might prescribe the velocity of a particle corresponding to a given piston motion which does not differ much from that which maintains a progressing wave. This would involve extensive computations although it can be reduced to a second order problem.

Alternately and more naturally, we may prescribe the total energy contained in the shock wave, that is, the energy imparted at $t = 0$. This is essentially equivalent to prescribing one condition on the t-axis, namely that no energy is added. From the general theory of hyperbolic equations we will have the correct number of conditions to determine the problem. In fact this problem can be solved in terms of the solutions of ordinary differential equations. In the case of a shock wave in water it can be solved explicitly.

Shock Wave of Constant Energy.

First we note that equations (11) with the boundary conditions (25) have the special solutions,

$$
\begin{aligned}
U_1 &= t^{2-2a}(\chi_{11}(\eta) + h\chi_{12}(\eta)) \\
C_1 &= t^{2-2a}(\chi_{21}(\eta) + h\chi_{22}(\eta)) \\
P_1 &= t^{2-2a}(\chi_{31}(\eta) + h\chi_{32}(\eta)) \\
H_1 &= ht^{2-2a}
\end{aligned}
\tag{27}
$$

where h is an arbitrary constant and χ_{11}, χ_{21}, χ_{31} is the solution of

$$
\begin{aligned}
\eta\frac{d\chi_1}{d\eta} &= \left(\frac{A}{D}\right)^o_U \chi_1 + \left(\frac{A}{D}\right)^o_C C_o\chi_2 - 2(1-a)D_o^{-1}(1-U_o)\chi_2 \\
&\quad - 2\gamma^{-1}(1-a)D_o^{-1}C_o^2\chi_3 \\
\eta\frac{d\chi_2}{d\eta} &= \left(\frac{B}{D}\right)^o_U \chi_1 + \left(\frac{B}{D}\right)^o_C C_o\chi_2 - (\gamma-1)(1-a)D_o^{-1}\chi_1 \\
&\quad - \frac{2}{1-U_o}\,\frac{1-a}{C_o}\,\chi_2 - \frac{\gamma-1}{\gamma}\,\frac{C_o^2 D_o^{-1}}{1-U_o}\,(1-a)\chi_3 \\
\eta\frac{d\chi_3}{d\eta} &= \left(\frac{E}{D}\right)^o_U \chi_1 + \left(\frac{E}{D}\right)^o_C C_o\chi_2 - 2\gamma(1-a)D_o^{-1}\chi_1 \\
&\quad - 2(1-a)D_o^{-1}(1-U_o)\chi_3
\end{aligned}
\tag{28}
$$

which satisfies the initial conditions

$$\chi_{11}(H_o) = -(1-\mu^2)$$

(29) $$\chi_{21}(H_o) = \frac{1-2\mu^4}{2\mu^2(1+\mu^2)}$$

$$\chi_{31}(H_o) = \frac{-\mu^2}{1+\mu^2}$$

and χ_{12}, χ_{22}, χ_{32} is the solution of (28) which satisfies the initial conditions

$$\chi_{12}(H_o) = -\left(\frac{A_o}{D_o}\right)_{\eta=H_o} - 2(1-\mu^2)(1-a)$$

(30) $$\chi_{22}(H_o) = -\left(\frac{B_o}{D_o}\right)_{\eta=H_o} - 2(1-a)$$

$$\chi_{32}(H_o) = -\left(\frac{E_o}{D_o}\right)_{\eta=H_o} - 4(1-a) \quad .$$

A method for reducing the third order system (28) to one of second order is contained in the Appendix.

It turns out that the solution (27) is just the solution which satisfies the condition of constant energy if h is chosen appropriately.

Let the total energy contained in a spherical shock wave at any time be E(t) where

(31) $$E(t) = 4\pi \int_0^{R(t)} \left(\frac{1}{2}\rho u^2 + \frac{p-p_1}{\gamma-1}\right) r^2\, dr \quad .$$

Here **R(t)** is the position of the shock. In terms of the variables (2) and (3), (31) becomes

(32) $$E(t) = -4\pi a^3 \int_\infty^{H(t)} \left(\frac{1}{2}\frac{\gamma U^2}{C^2} + \frac{1}{\gamma-1}\right) P\, \eta^{-5a-1}\, t^{5a-2}\, d\eta - \frac{4\pi}{3}\frac{p_1}{\gamma-1} R^3(t)$$

where $\eta = H(t)$ represents the shock.

In the case of an infinitely strong shock, where $p_1 = 0$, $E(t)$ will be constant only if $\alpha = 2/5$. Then

$$(33) \qquad E = E_o = \frac{-32\pi}{125} \int_{\infty}^{H(t)} \left(\frac{1}{2} \frac{\gamma U_o^2}{C_o^2} + \frac{1}{\gamma-1} \right) P_o \eta^{-3} \, d\eta \quad .$$

If we now consider that second order terms in p_1 can be neglected and use the approximations (10) and (19) we obtain

$$(34) \qquad E = -\frac{32\pi}{125} \int_{\infty}^{H_o} \left(\frac{1}{2} \frac{\gamma U_o^2}{C_o^2} + \frac{1}{\gamma-1} \right) P_o \eta^{-3} \, d\eta$$

$$- \frac{32\pi}{125} \varepsilon H_o H_1 \left[\left(\frac{1}{2} \frac{\gamma U_o^2}{C_o^2} + \frac{1}{\gamma-1} \right) P_o \eta^{-3} \right]_{\eta = H_o}$$

$$- \frac{32\pi\varepsilon}{125} \int_{\infty}^{H_o} \left\{ \left(\frac{\gamma}{2} \frac{U_o^2}{C_o^2} + \frac{1}{\gamma-1} \right) P_o P_1 + \frac{\gamma U_o P_o}{C_o^2} (U_1 - U_o C_1) \right\} \eta^{-3} \, d\eta + O(\varepsilon^2)$$

$$= E_o + \frac{4\pi p_1}{3(\gamma-1)} H_o^{-6/5} t^{6/5} \quad .$$

From the first order terms in ε using, from (21),

$$\varepsilon = \lambda^2 c_1^2 H_o^{2\alpha} = \frac{25}{4} \gamma p_1 \rho_1^{-1} H_o^{4/5}$$

we obtain

$$(35) \qquad H_o^{-2} H_1 \left[\left(\frac{1}{2} \frac{\gamma U_o^2}{C_o^2} + \frac{1}{\gamma-1} \right) P_o \right]_{\eta = H_o} + \int_{\infty}^{H_o} \left\{ \left(\frac{\gamma}{2} \frac{U_o^2}{C_o^2} + \frac{1}{\gamma-1} \right) P_o P_1 + \frac{\gamma U_o P_o}{C_o^2} (U_1 - U_o C_1) \right\} \eta^{-3} \, d\eta$$

$$= -\frac{5}{2} \frac{\rho_1}{\gamma(\gamma-1)} t^{6/5} H_o^{-2}$$

or, using (24),

$$(36)\quad H_1\,\frac{2(1-\mu^2)}{\gamma-1}+\int_\infty^1\left\{\left(\frac{\gamma}{2}\frac{U_o^2}{C_o^2}+\frac{1}{\gamma-1}\right)P_1+\frac{\gamma U_o}{C_o^2}(U_1-U_oC_1)\right\}\frac{P_o}{\rho_1}\left(\frac{\eta}{H_o}\right)^{-3}d\left(\frac{\eta}{H_o}\right)=-\frac{5}{2}\frac{1}{\gamma(\gamma-1)}\,t^{6/5}\;.$$

We can now show that (27) is just the solution of (11) and (25) which satisfies (36) if h is chosen appropriately.

Substituting (27) in (36) using $r^{-\lambda}t = H_o$ to first order on the shock yields

$$ht^{6/5}\left[\frac{2(1-\mu^2)}{\gamma-1}+\int_\infty^1\left\{\left(\frac{\gamma}{2}\frac{U_o^2}{C_o^2}+\frac{1}{\gamma-1}\right)\chi_{32}+\frac{\gamma U_o}{C_o^2}(\chi_{12}-U_o\,\chi_{22})\right\}\frac{P_o}{\rho_1}\left(\frac{\eta}{H_o}\right)^{-21/5}d\left(\frac{\eta}{H_o}\right)\right]+t^{6/5}\int_\infty^1\left\{\left(\frac{\gamma}{2}\frac{U_o^2}{C_o^2}+\frac{1}{\gamma-1}\right)\chi_{31}+\frac{\gamma U_o}{C_o^2}(\chi_{11}-U_o\,\lambda_{21})\right\}\frac{P_o}{\rho_1}\left(\frac{\eta}{H_o}\right)^{-21/5}d\left(\frac{\eta}{H_o}\right)=-\frac{5}{2}\frac{1}{\gamma(\gamma-1)}\,t^{6/5}$$

or

$$(37)\qquad A_1h+A_2=A_3$$

where

$$A_1=\frac{2(1-\mu^2)}{\gamma-1}+\int_\infty^1\left\{\left(\frac{\gamma}{2}\frac{U_o^2}{C_o^2}+\frac{1}{\gamma-1}\right)\chi_{32}+\frac{\gamma U_o}{C_o^2}(\chi_{12}-U_o\,\chi_{22})\right\}\frac{P_o}{\rho_1}\left(\frac{\eta}{H_o}\right)^{-21/5}d\left(\frac{\eta}{H_o}\right)$$

$$(38)\quad A_2=\int_\infty^1\left\{\left(\frac{\gamma}{2}\frac{U_o^2}{C_o^2}+\frac{1}{\gamma-1}\right)\chi_{31}+\frac{\gamma U_o}{C_o^2}(\chi_{11}-U_o\,\chi_{21})\right\}\frac{P_o}{\rho_1}\left(\frac{\eta}{H_o}\right)^{-21/5}d\left(\frac{\eta}{H_o}\right)$$

$$A_3=-\frac{5}{2}\frac{1}{\gamma(\gamma-1)}\;.$$

Equation (37) can always be solved for h provided $A_1 \neq 0$. Thus the first order increment in the energy vanishes and we have a complete solution of the problem in terms of χ_{11} etc.

From (7), (9) and (24) we see that U_o and C_o and P_o/ρ_1 are functions of η/H_o only and independent of ρ_1 while P/ρ_1 is a function of η/H_o.

Thus the coefficients of the equations (28) are functions of η/H_o and hence by (28), (29) and (30) the functions χ_{11}, χ_{21}, χ_{31}, χ_{12}, χ_{22}, χ_{32} depend on η/H_o only.

Thus A_1 and A_2 are constants depending only on the solutions of fixed differential equations with fixed initial conditions, depending only on γ.

Substituting (27) in (10) and then in (3) we find

$$
\begin{aligned}
u &= \frac{2}{5}\frac{r}{t}\left(U_o + \varepsilon t^{6/5}(\chi_{11} + h\chi_{12})\right) \\
&= \frac{2}{5}\, t^{-3/5}\eta^{-2/5}U_o + \frac{2}{5}\,\varepsilon t^{3/5}\eta^{-2/5}(\chi_{11} + h\chi_{12}) \\
c &= \frac{2}{5}\, t^{-3/5}\eta^{-2/5}C_o + \frac{2}{5}\,\varepsilon t^{3/5}\eta^{-2/5}C_o(\chi_{21} + h\chi_{22}) \\
p &= \frac{4}{25}\, t^{-6/5}\eta^{-4/5}P_o + \frac{4}{25}\,\varepsilon\,\eta^{-4/5}P_o(\chi_{31} + h\chi_{32}) \quad .
\end{aligned}
$$

Note that for fixed η the perturbations in u and c increase with r while the perturbation in p is constant.

Now the pressure dies out behind the shock and the maximum pressure occurs at the shock; thus from (22) and (3) we find

$$
p_{max} = \frac{4}{25} H_o^{-4/5}\rho_1(1-\mu^2)t^{-6/5} - \left(\frac{12}{5}h + \frac{\mu^2}{1+\mu^2}\right)(1-\mu^2)P_1
$$

In other words, the maximum pressure is the pressure of the Taylor point blast wave plus a constant.

Shockwave of Constant Energy in Water.

In the case of an explosion in water the functions χ_{11} etc. can be calculated explicitly.

Here we have $P = A((\frac{\rho}{\rho_o})^{\gamma} - 1)$ where $A = 3000$ atmospheres and $\gamma = 7$. Then we replace the last equation of (3) by $P = \lambda^2 t^2 r^{-2}(p + A)$. In this case,

$$(39) \quad U_o \equiv 1-\mu^2 \ , \quad C_o^2 \equiv \mu^2(1+\mu^2) \ , \quad P_o \equiv \rho_1(1-\mu^2)\left(\frac{\eta}{H_o}\right)^{-2/5}$$

is the solution of (9) and (24)*. Then $A_o = 0$, $B_o = 0$ while E_o and D_o are constants.

The differential equations (28) then have constant coefficients in the lowest order terms and the solutions are powers of η. Satisfying equations (29) and (30) we find,

$$(40) \quad \chi_{11} = -1.00197\left(\frac{\eta}{H_o}\right)^{5.4788} - .5605\left(\frac{\eta}{H_o}\right)^{0.3212}$$

$$\chi_{21} = 2.2689\left(\frac{\eta}{H_o}\right)^{-1.6000} - 5.4262\left(\frac{\eta}{H_o}\right)^{5.4788} + 2.8597\left(\frac{\eta}{H_o}\right)^{0.3212}$$

$$\chi_{31} = -1.5126\left(\frac{\eta}{H_o}\right)^{-1.6000} - 2.3471\left(\frac{\eta}{H_o}\right)^{5.4788} + 1.1811\left(\frac{\eta}{H_o}\right)^{0.3212}$$

$$\chi_{12} = -0.3885\left(\frac{\eta}{H_o}\right)^{5.4788} + 0.0885\left(\frac{\eta}{H_o}\right)^{0.3212}$$

$$\chi_{22} = 1.3553\left(\frac{\eta}{H_o}\right)^{-1.6000} - 2.1039\left(\frac{\eta}{H_o}\right)^{5.4788} - 0.4514\left(\frac{\eta}{H_o}\right)^{0.3212}$$

$$\chi_{23} = -0.9035\left(\frac{\eta}{H_o}\right)^{-1.6000} - 0.9100\left(\frac{\eta}{H_o}\right)^{5.4788} - 0.1865\left(\frac{\eta}{H_o}\right)^{0.3212}$$

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* This special solution is due to H. Primakoff.

Substituting (39) and (40) in (38) and then (37) we find that

$$h = -3.4938$$

and finally, from (27),

$$(41) \quad U_1 = t^{6/5}\left\{.3553\left(\frac{\eta}{H_o}\right)^{5.4788} -.8697\left(\frac{\eta}{H_o}\right)^{0.3212}\right\}$$

$$C_1 = r^3\left\{-1.6442\left(\frac{\eta}{H_o}\right)^{-1.6000} +.8323\left(\frac{\eta}{H_o}\right)^{5.4788} +1.8326\left(\frac{\eta}{H_o}\right)^{0.3212}\right\}$$

$$P_1 = r^3\left\{-2.4662\left(\frac{\eta}{H_o}\right)^{-1.6000} +1.9242\left(\frac{\eta}{H_o}\right)^{5.4788} +4.4369\left(\frac{\eta}{H_o}\right)^{0.3212}\right\}$$

$$H_1 = -3.4938\ t^{6/5} \quad .$$

In Figures 1, 2 and 3 we have plotted the path of the shock as a function of time and the maximum pressure as a function of time and distance.

State Behind a Detonation.

The third shock condition in (12) must be modified to include the chemical energy of the detonation. We then have for the conditions across the front

$$(42) \qquad \rho(u-z) = -\rho_1 z$$

$$\rho(u-z)^2 + p = \rho_1 z^2 + p_1$$

$$\frac{\gamma p}{\rho(\gamma-1)} + \frac{1}{2}(u-z)^2 + \bar{E} = \frac{z^2}{2} + \frac{\gamma p_1}{\rho_1(\gamma_1-1)} + \bar{E}_1$$

where $\bar{E}$ and $\bar{E}_1$ are the energy of formation per unit mass of the burnt and unburnt material respectively, and γ_1 is the ratio of the specific heats in the unburnt gas. In terms of the variables (3) and (14) the last condition of (42) becomes

$$\mu^2(U-Z)^2 + (1-\mu^2)C^2 = \mu^2 Z^2 + \left[\frac{\mu^2}{\mu_o^2}(1-\mu_o^2)C_o^2 + (\bar{E} - \bar{E}_1)\right]\lambda^2 \; r^{2\lambda-2}\eta^2 \quad .$$

If we perturb about a strong detonation, i.e. setting $\frac{\mu^2}{\mu_o^2}(1-\mu_o^2)C_o^2 + \bar{E} - \bar{E}_1 = 0$ we can find the undisturbed flow as a spherical wave and the perturbed flow satisfies (25) with different constants for the coefficients of $r^{2\lambda-2}$.

In this case there are again special solutions of the form (27) where now the initial conditions (29) must be adjusted appropriately. However, in this case we need one more condition on the flow and the special solution will not in general satisfy it.

For $\gamma = 2$ we have $U_o(H_o) = C_o(H_o)$ or $u = c$ at the shock. The unperturbed detonation is then a Chapman-Jouguet detonation. If the perturbed flow is also behind a Chapman-Jouguet detonation we obtain a relation between $U_1(H_o)$, $C_1(H_o)$ and $H_1(r)$. This relation can be satisfied by solutions of the form (27) for an appropriate choice of h.

Appendix.

In this Appendix we will show how equations (28) can be reduced to one second order equation. In general, equations (6) and the shock conditions (25) can be reduced to a second order equation involving the unknown shock function $H_1(x)$.

From equations (8) we see that

$$\text{(A.1)} \qquad \left(\frac{2}{\gamma-1}\right)\left(\frac{B}{D}\right)_C^o - \left(\frac{A}{D}\right)_C^o \frac{1}{1-U_o} = 0$$

$$\gamma^{-1}\left(\frac{E}{D}\right)_C^o - \left(\frac{A}{D}\right)_C^o \frac{1}{1-U_o} = 0$$

and from (11) then,

$$(A.2)\quad \frac{2}{\gamma-1}\eta\frac{d\chi_2}{d\eta}-\frac{1}{1-U_o}\eta\frac{d\chi_1}{d\eta}$$

$$=\left(\frac{2}{\gamma-1}\left(\frac{B}{D}\right)_U^o-\frac{1}{1-U_o}\left(\frac{A}{D}\right)_U^o\right)\chi_1-\frac{4}{1-U_o}\frac{(1-\alpha)}{\gamma-1}C_o^{-1}\chi_2$$

$$\gamma^{-1}\eta\frac{d\chi_3}{d\eta}-\frac{1}{1-U_o}\eta\frac{d\chi_1}{d\eta}$$

$$=\left(\gamma^{-1}\left(\frac{E}{D}\right)_U^o-\frac{1}{1-U_o}\left(\frac{A}{D}\right)_U^c\right)\chi_1-\gamma^{-1}\frac{2(1-\alpha)}{1-U_o}\chi_3$$

$$\eta\frac{d\chi_1}{d\eta}=\left(\left(\frac{A}{D}\right)_U^o+2(1-U_o)(1-\alpha)D_o^{-1}\right)\chi_1$$

$$+\left(\frac{A}{D}\right)_U^o C_o\chi_2-2\gamma^{-1}C_o^2D_o^{-1}(1-\alpha)\chi_3\quad .$$

If we introduce new dependent variables,

$$(A.3)\qquad \xi_1(\eta)=-\chi_1(\eta)/U_o(1-U_o)$$

$$\xi_2(\eta)=\frac{2}{\gamma-1}\chi_2(\eta)+\xi_1(\eta)U_o$$

$$\xi_3(\eta)=\gamma^{-1}\chi_3(\eta)+\xi_1(\eta)U_o$$

equations (A.2) reduce to,

$$(A.4)\quad (1-U_o)\eta\frac{d\xi_1}{d\eta}$$

$$=\lambda V_J\left[\frac{2U_o-1}{U_o^2}\frac{A_o}{D_o}+\frac{1-U_o}{U_o}\left(\frac{A}{D}\right)_U^o+\frac{\gamma-1}{2}\frac{C_o}{U_o^2}\left(\frac{A}{D}\right)_C^o-2\left(1+\frac{2C_o^2}{U_oD_o}\right)(1-\alpha)\alpha\right]$$

$$-\left(\frac{A}{D}\right)_C^o\frac{C_o}{U_o^2}\frac{\gamma-1}{2}\xi_2+\frac{2C_o^2}{U_o^2D_o}(1-\alpha)\xi_3$$

$$\frac{1-U_o}{U_o}\,\eta\,\frac{d\xi_2}{d\eta} = (-3 + \frac{2}{\gamma-1}(\lambda-1) + 2(1-\alpha))\xi_1 - (2-2\alpha)\xi_2$$

$$\frac{1-U_o}{U_o}\,\eta\,\frac{d\xi_3}{d\eta} = (-3 + 2\gamma^{-1}(\lambda-1) + 2(1-\alpha))\xi_1 - U_o^{-1}(2-2\alpha)\xi_3 \quad .$$

By introducing new dependent variables we can reduce (A.4) to a much simpler form. We set

(A.5) $$y = \int_{H_*}^{\eta} \frac{U_o}{1-U_o}\frac{d\eta}{\eta}$$

where H_* is any fixed value of η. Then (A.4) reduces to

(A.6) $$D_o\frac{d\xi_1}{dy} + K_o\xi_1 + L_o\xi_2 + 2(1-\alpha)\frac{C_o^2}{U_o^2}\xi_3 = 0$$

$$\xi_{2y} - \mu_2\xi_1 = 0$$

$$\xi_{3y} - \mu_3\xi_1 = 0$$

where K_o and L_o are functions of y,

(A.7) $$K_o(y) = -\lambda\left\{\frac{2U_o-1}{U_o^2}A_o + \frac{1-U_o}{U_o}D_o\left(\frac{A}{D}\right)_U^o + \frac{\gamma-1}{2}\frac{C_o}{U_o^2}D_o\left(\frac{A}{D}\right)_C^o\right\} - 4(1-\alpha)\frac{C_o^2}{U_o}$$

$$L_o(y) = \lambda\,\frac{\gamma-1}{2}\,\frac{C_o}{U_o^2}\left(\frac{A}{D}\right)_C^o$$

and μ_2 and μ_3 are constants,

(A.8) $$\mu_2 = 3 - 2\gamma^{-1}\lambda + 2\gamma^{-1} - 2(1-\alpha)$$

$$\mu_3 = 3 - \frac{2}{\gamma-1}\lambda + \frac{2}{\gamma-1} - 2(1-\alpha) \quad .$$

The solutions of the third order system (A.6) can also be expressed in terms of the solutions of a second order differential equation. We introduce the function G where

(A.9) $$\frac{d}{dy} G = \xi_1$$

(A.10) $$G(Y_o) = 0$$

where Y_o is some fixed value of y.

Then from the last equations (A.6) we obtain

(A.11) $$\xi_2(y) = \xi_2(Y_o) + \mu_2 G(y)$$

(A.12) $$\xi_3(y) = \xi_3(Y_o) + \mu_3 G(y) \quad .$$

Substituting (A.11) and (A.12) in the first equation of (A.6) yields

(A.13) $$D_o \frac{d^2G}{dy^2} + K_o \frac{dG}{dy} + G\left(\mu_2 L_o + (2-2\alpha)\mu_3 \frac{C_o^2}{U_o^2}\right)$$

$$= -L_o\xi_2(Y_o) - 2(1-\alpha) \frac{C_o^2}{U_o^2} \xi_3(Y_o) \quad .$$

Now ξ_2 and ξ_3 are prescribed for $y = Y_o$ by (29) or (30). Thus we have a differential equation (A.13) for G and two boundary conditions (A.9) and (A.10) at $y = Y_o$.

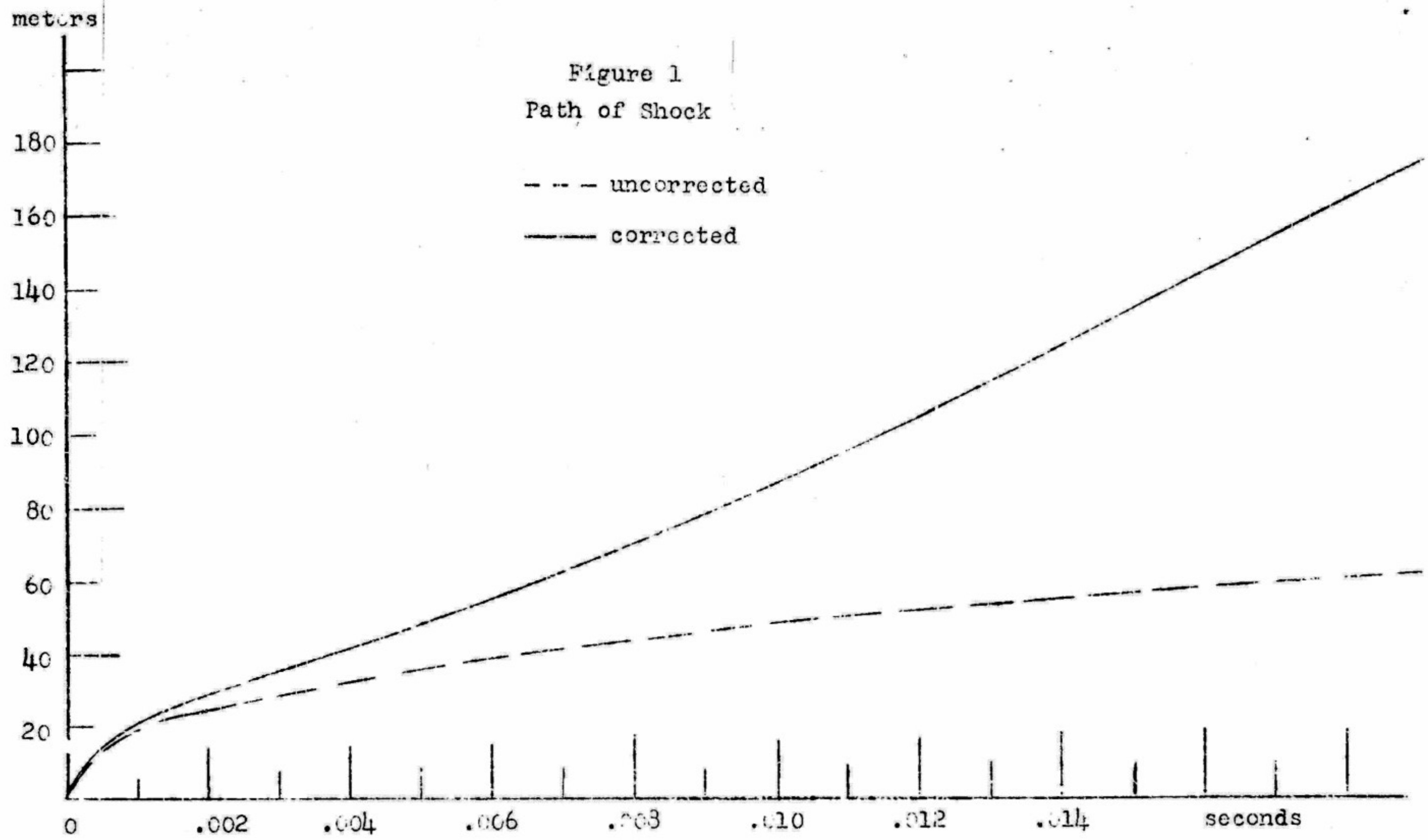

Figure 1
Path of Shock

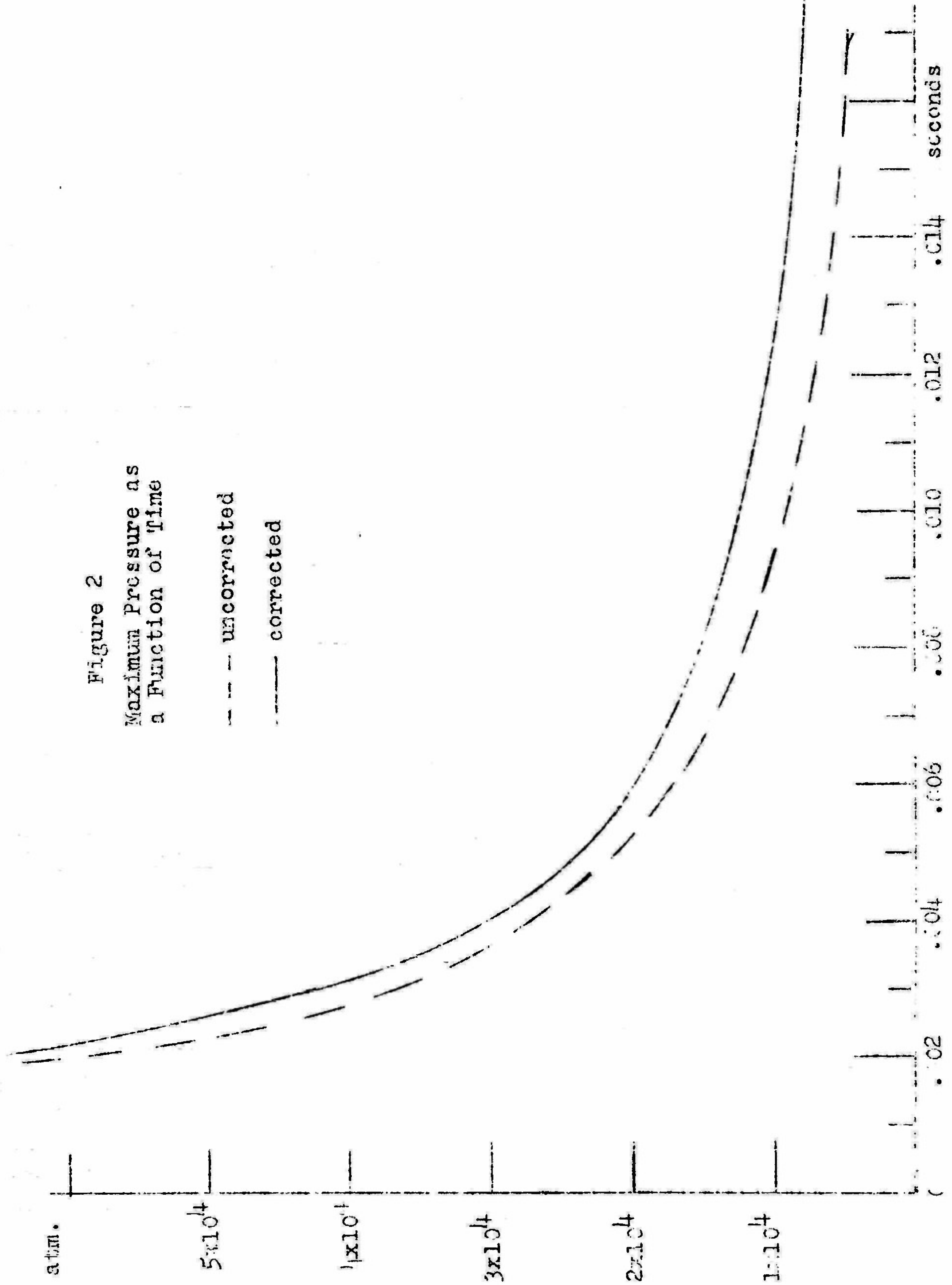

Figure 2
Maximum Pressure as a Function of Time

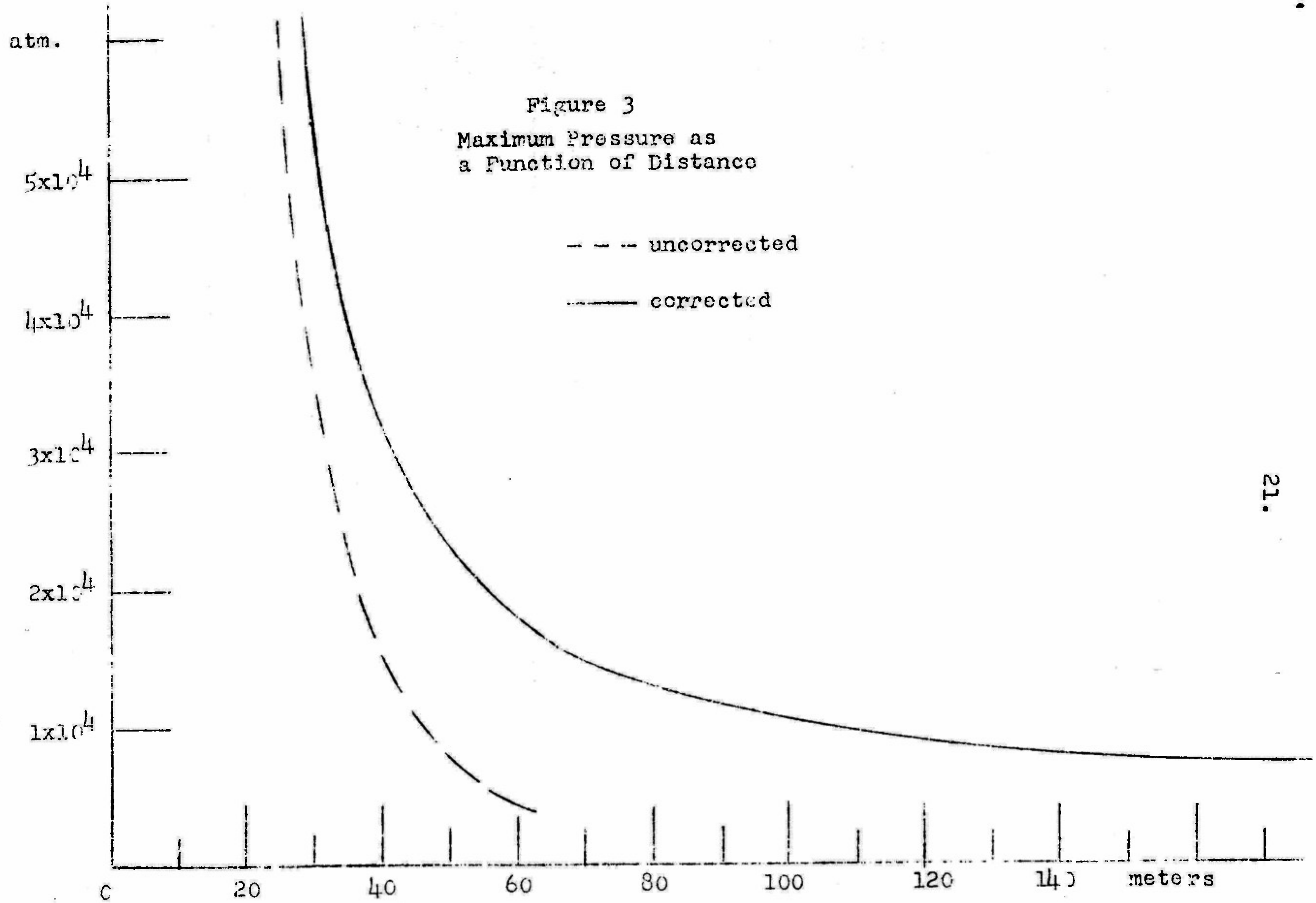

Figure 3
Maximum Pressure as a Function of Distance